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Norm-Preserving Extension of Convex Lipschitz Functions

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Let (X, d) be a metric space. A function $f: X \to R$ is called *Lipschitz* if there exists a number $M \geqslant 0$ such that

$$|f(x) - f(y)| \leqslant Md(x, y) \tag{1}$$

for all $x, y \in X$. The smallest constant M verifying (1) is called the *norm* of f and is denoted by $\|f\|_X$.

We have

$$\|f\|_X = \sup\{|f(x) - f(y)|/d(x, y) : x, y \in X, x \neq y\}. \tag{2}$$

Denote by Lip X the linear space of all Lipschitz functions on X. Actually, $\|\cdot\|_X$ is not a norm on the space Lip X, since $\|f\|_X = 0$ if f is constant.

Now let Y be a nonvoid subset of X. A *norm-preserving extension* of a function $f \in \text{Lip } Y$ to X is a function $F \in \text{Lip } X$ such that $F|_Y = f$ and $\|f\|_Y = \|F\|_X$. By a result of Banach [1] (see also Czipser and Geher [2]) every $f \in \text{Lip } Y$ has a norm-preserving extension F in Lip X. Two of these extensions are given by

$$F_1(x) = \sup\{f(y) - \|f\|_Y\, d(x, y) : y \in Y\} \tag{3}$$

and

$$F_2(x) = \inf\{f(y) + \|f\|_Y\, d(x, y) : y \in Y\}. \tag{4}$$

Every norm-preserving extension F of f satisfies

$$F_1(x) \leqslant F(x) \leqslant F_2(x) \tag{5}$$

for all $x \in X$ (see [7]).

Now, let X be a normed linear space and let Y be a nonvoid convex subset of X. Concerning the convex norm-preserving extension to X of the convex functions in Lip Y, we can prove the following theorem:

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THEOREM 1. *If X is a normed linear space and Y a nonvoid convex subset of X, then every convex function f in* Lip Y *has a convex norm preserving extension F in* Lip X.

Proof. We show that the maximal norm-preserving extension (4) of f is also convex. Let $F(x) = \inf\{f(y) + \|f\|_Y \| x - y \| : y \in Y\}$, $x_1, x_2 \in X$, y_1, $y_2 \in Y$, and $\alpha \in [0, 1]$. Then

$$\begin{aligned} &F(\alpha x_1 + (1 - \alpha) x_2) \\ &\quad \leqslant f(\alpha y_1 + (1 - \alpha) y_2) + \|f\|_Y \| \alpha x_1 + (1 - \alpha) x_2 - \alpha y_1 - (1 - \alpha) y_2 \| \\ &\quad \leqslant \alpha f(y_1) + (1 - \alpha) f(y_2) + \|f\|_Y(\alpha \| x_1 - y_1 \| + (1 - \alpha)\| x_2 - y_2 \|) \\ &\quad = \alpha(f(y_1) + \|f\|_Y \| x_1 - y_1 \|) + (1 - \alpha)(f(y_2) + \|f\|_Y \| x_2 - y_2 \|). \end{aligned}$$

Taking the infimum with respect to y_1, $y_2 \in Y$, we obtain

$$F(\alpha x_1 + (1 - \alpha) x_2) \leqslant \alpha F(x_1) + (1 - \alpha) F(x_2),$$

which shows that the function F is convex.

In general, this extension is not unique. Indeed, let $X = R$, with the usual absolute value norm, $Y = [-1, 1]$, and f: $Y \to R$ be given by $f(x) = -x$ for $x \in [-1, 0]$ and $f(x) = 2x$ for $x \in]0, 1]$. Then the maximal norm-preserving extension (4) of f is given by $F(x) = -2x$ for $x \in]-\infty, -1[$, $F(x) = -2x$ for $x \in [-1, 0[$, and $F(x) = 2x$ for $x \in [0, +\infty[$. But the function $G(x) = -x$ for $x \in]-\infty, 0[$ and $G(x) = 2x$ for $x \in [0, +\infty[$ is also a convex norm-preserving extension of f, and so is every convex combination $\alpha F + (1 - \alpha) G$, $\alpha \in [0, 1]$, of the functions F and G.

Let, as above, X be a normed linear space and Z a convex subset of X such that $0 \in Z$. Denote by $\mathrm{Lip}_0\, Z$ the space

$$\mathrm{Lip}_0\, Z = \{f \in \mathrm{Lip}\, Z : f(0) = 0\}. \tag{6}$$

Then (2) is a norm on $\mathrm{Lip}_0\, Z$ and $\mathrm{Lip}_0\, Z$ is a Banach space with respect to this norm.

We use also the following notations:

$$K_Z = \{f \in \mathrm{Lip}_0\, Z : f \text{ is convex on } Z\}, \tag{7}$$

—the convex cone of convex functions in $\mathrm{Lip}_0\, Z$;

$$X_c = K_X - K_X, \tag{8}$$

—the linear space generated by the cone K_X;

$$Z_c^{\perp} = \{f \in X_c : f|_Z = 0\}, \tag{9}$$

—the null space of the set Z in X_c.

If E is a normed linear space, M a nonvoid subset of E and $x \in E$, we denote by $d(x, M)$ the *distance* from x to M, i.e.,

$$d(x, M) = \inf\{\|x - y\| : y \in M\}$$

and by P_M the *metric projection* of X onto M, i.e.,

$$P_M(x) = \{y \in M : \|x - y\| = d(x, M)\}.$$

If K is a subset of X, then the set M is called *K-proximinal* (*K-Chebyshevian*) if $P_M(x) \neq \varnothing$ (respectively card$(P_M(x)) = 1$), for all $x \in K$.

In the sequel X denotes a normed linear space and Y a convex subset of X such that $0 \in Y$. It follows that K_Y is a P-cone in the sense of [10], and as a particular case of the results proved there, one obtains:

THEOREM 2. (a) *If $f \in K_X$ then*

$$\| f|_Y \|_Y = d(f, Y_c^{\perp}).$$

(b) *The space $Y_c^{\perp}$ is K_X-proximinal. For $f \in K_X$, the function g is in $P_{Y_c^{\perp}}(f)$ if and only if $g = f - F$, where F is a convex norm-preserving extension of $f|_Y$.*

(c) *The space $Y_c^{\perp}$ is K_X-Chebyshevian if and only if every $f \in K_Y$ has a unique convex norm-preserving extension to X.*

Remark. Similar duality results appear in [4, 11] for linear functionals and in [6–10] for Lipschitz functions.

Now, we want to show that an inequality similar to (5) holds also for the convex norm-preserving extensions of a given convex Lipschitz function. For $f \in K_Y$ let us denote by $E_Y^c(f)$ the set of all convex norm preserving extensions of f. We denote the norm $\|\cdot\|_X$ by $\|\cdot\|$.

THEOREM 3. *If $f \in K_Y$ then there exist two functions F_1, F_2 in $E_Y^c(f)$ such that*

$$F_1(x) \leqslant F(x) \leqslant F_2(x) \tag{10}$$

for all $x \in X$ and $F \in E_Y^c(f)$.

For the proof we need the following lemma:

LEMMA 4. *The set $E_Y^c(f)$ is downward directed (with respect to the pointwise ordering).*

Proof of Lemma 4. We have to show that for $G_1, G_2 \in E_Y{}^c(f)$ there exists $G \in E_Y{}^c(f)$ such that

$$G(x) \leqslant \min(G_1(x), G_2(x)), \tag{11}$$

for all $x \in X$.

If E is a linear space and $\varphi : E \to R \cup \{\pm\infty\}$ is a function, then the strict epigraph of φ is defined by

$$\text{epi}'\, \varphi = \{(x, a) \in E \times R : \varphi(x) < a\}.$$

The function φ is convex if and only if its strict epigraph is a convex subset of $E \times R$ (see Laurent [5, Theorem 6.1.5, Remark 6.1.6]).

For $G_1, G_2 \in E_Y{}^c(f)$ put

$$\Gamma = \text{co}(\text{epi}'\, G_1 \cup \text{epi}'\, G_2), \tag{12}$$

where co(A) denotes the convex hull of the set A.

Define $G: X \to R \cup \{\pm\infty\}$ by

$$G(x) = \inf\{a \in R : (x, a) \in \Gamma\}, \qquad x \in X. \tag{13}$$

We show that $G \in E_Y{}^c(f)$ and that G verifies the inequality (11). The proof is divided into several steps.

(i) *The set Γ is open.* Since the functions G_1 and G_2 are continuous, the sets epi′ G_1 and epi′ G_2 are open, and so is their convex hull Γ.

(ii) *If $(z, c) \in \Gamma$ and $d \geqslant c$ then $(z, d) \in \Gamma$.* Let $z = \alpha x + (1 - \alpha) y$, $c = \alpha a + (1 - \alpha) b$, for $\alpha \in [0, 1]$, $(x, a) \in \text{epi}'\, G_1$, $(y, b) \in \text{epi}'\, G_2$ and let $\epsilon > 0$ be an arbitrary number. Then $(x, a + \epsilon) \in \text{epi}'\, G_1$ and $(y, b + \epsilon) \in \text{epi}'\, G_2$, so that $(z, c + \epsilon) = \alpha(x, a + \epsilon) + (1 - \alpha)(y, b + \epsilon) \in \Gamma$.

(iii) *epi′ $G = \Gamma$ and G is a convex function.* Let $(x, a) \in \text{epi}'\, G$, i.e., $G(x) < a$. By (13) there exists $b \in R$ such that $(x, b) \in \Gamma$ and $b < a$. By (ii), $(x, a) \in \Gamma$, proving the inclusion epi′ $G \subset \Gamma$.

Conversely, let $(x, a) \in \Gamma$. By (i) Γ is open, so that there exist a neighborhood U of x and $\epsilon > 0$ such that $U \times]a - \epsilon, a + \epsilon[\subset \Gamma$. Therefore $\{x\} \times]a - \epsilon, a + \epsilon[\subset \Gamma$ and, by (13), $G(x) \leqslant a - \epsilon < a$, which shows that $(x, a) \in \text{epi}'\, G$ and $\Gamma \subset \text{epi}'\, G$.

The convexity of G follows from the above quoted result in Laurent [5].

(iv) *We have $G(x) \leqslant \min(G_1(x), G_2(x))$ for all $x \in X$ and $G(z) = G_1(z) = G_2(z)$ for all $z \in Y$.* Let $x \in X$. Then for all $a > G_1(x)$ and $b > G_2(x)$ we have $(x, a) \in \text{epi}'\, G_1 \subset \Gamma$ and $(y, b) \in \text{epi}'\, G_2 \subset \Gamma$, so that, by (13), $G(x) \leqslant \min(G_1(x), G_2(x))$.

Let z be in Y and c in R such that $(z, c) \in \Gamma$. Then $(z, c) = \alpha(x, a) + (1 - \alpha)(y, b)$, for a number $\alpha \in [0, 1]$, $(x, a) \in \text{epi}' G_1$, and $(y, b) \in \text{epi}' G_2$. But, by the convexity of G_1 and G_2, $G_i(z) = G_i(\alpha x + (1 - \alpha) y) \leqslant \alpha G_i(x) + (1 - \alpha) G_i(y) < \alpha a + (1 - \alpha) b = c$, for $i = 1, 2$. Taking the infimum with respect to all $c \in R$ such that $(z, c) \in \Gamma$ we obtain $G(z) \geqslant G_1(z) = G_2(z)$. Since the converse inequality holds for all $x \in X$, it follows $G(z) = G_1(z) = G_2(z)$, for all $z \in Y$.

(v) $-\infty < G(x) < +\infty$ *for all* $x \in X$. The relations $(x, G_1(x) - 1) \in \text{epi}' G_1 \subset \Gamma$ and (13) imply $G(x) \leqslant G_1(x) + 1 < \infty$. Suppose there exists $x \in X$ such that $G(x) = -\infty$. Choose an element $y \in Y$ and put $z = 2y - x$. Then, by (iv) and the convexity of G we get

$$G_1(y) = G(y) \leqslant 2^{-1}(F(x) + F(z)) = -\infty,$$

implying $G_1(y) = -\infty$, which is impossible.

(vi) *Equality of the norms*: $\| G \| = \| f \|_Y = \| G_1 \| = \| G_2 \|$. Since $G |_Y = G_1 |_Y = f$, it follows $\| G \| \geqslant \| G_1 \|$. Suppose $\| G \| > \| G_1 \|$. By the definition (2) of the norm in Lip X, there exist $x, y \in X$, $x \neq y$ such that $| G(x) - G(y)|/\| x - y \| > \| G_1 \|$, say

$$| G(x) - G(y)|/\| x - y \| = \| G_1 \| + \epsilon,$$

for an $\epsilon > 0$. Without loss of generality we can suppose

$$\frac{G(y) - G(x)}{\| x - y \|} = \| G_1 \| + \epsilon. \tag{14}$$

Let $\overrightarrow{xy} = \{x + t(y - x) : t \geqslant 0\}$ be the half-line determined by x and y. Define $\varphi :]0, \infty[\to R$ by $\varphi(t) = t^{-1}(G(x + t(y - x)) - G(x))$. By Holmes [3, p. 17], the function φ is nondecreasing, so that

$$\begin{aligned} \frac{G(x + t(y - x)) - G(x)}{\| t(y - x)\|} &= \frac{1}{\| y - x \|} \cdot \varphi(t) \geqslant \frac{1}{\| y - x \|} \cdot \varphi(1) \\ &= \frac{G(y) - G(x)}{\| y - x \|} = \| G_1 \| + \epsilon \\ &\geqslant \frac{G_1(x + t(y - x)) - G_1(x)}{\| t(y - x)\|} + \epsilon, \end{aligned}$$

for all $t \geqslant 1$.

Therefore

$$G_1(x + t(y - x)) \leqslant G(x + t(y - x)) - (G(x) - G_1(x) + t\epsilon \| y - x \|),$$

for all $t \geqslant 1$. But for t sufficiently large, $G(x) - G_1(x) + t\epsilon \| y - x \| > 0$, so that $G_1(x + t(y - x)) < G(x + t(y - x))$, contradicting the inequality $G \leqslant G_1$ (iv).

Lemma 4 is completely proved.

Proof of Theorem 3. Let F_2 be the maximal norm-preserving extension (4) of f. By the proof of Theorem 1, F_2 is convex and since $F_2(x) \geqslant F(x)$ for every norm-preserving extension F of f, this is a fortiori true for the convex norm-preserving extensions of f.

Put

$$F_1(x) = \inf\{F(x) : F \in E_Y{}^c(f)\}. \tag{15}$$

To end the proof we have to show that F_1 is a convex norm-preserving extension of f.

(i) *F_1 is a convex function.* Let $x, y \in X$, $\alpha \in [0, 1]$, $\epsilon > 0$ and let G_1, $G_2 \in E_Y{}^c(f)$ be such that $G_1(x) < F_1(x) + \epsilon$ and $G_2(y) < F_1(y) + \epsilon$. Since, by Lemma 4, the set $E_Y{}^c(f)$ is downward directed, there exists $G_3 \in E_Y{}^c(f)$ such that $G_3 \leqslant G_1$ and $G_3 \leqslant G_2$. Then

$$\begin{aligned} &F_1(\alpha x + (1 - \alpha) y) \\ &\quad\leqslant G_3(\alpha x + (1 - \alpha) y) \leqslant \alpha G_3(x) + (1 - \alpha) G_3(y) \\ &\quad\leqslant \alpha G_1(x) + (1 - \alpha) G_2(y) < \alpha F_1(x) + (1 - \alpha) F_2(y) + \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, it follows that

$$F_1(\alpha x + (1 - \alpha) y) \leqslant \alpha F_1(x) + (1 - \alpha) F_2(y),$$

i.e., the function F_1 is convex.

(ii) $F_1|_Y = f$. This is obvious since $F(y) = f(y)$ for all $y \in Y$ and $F \in E_Y{}^c(f)$.

(iii) *Equality of the norms*: $\| F_1 \| = \| f \|_Y$. Obviously, $\| F_1 \| \geqslant \| f \|_Y$. Let us suppose $\| F_1 \| > \| f \|_Y$. Then there exists $\delta > 0$ such that $\| F_1 \| = \| f \|_Y + \delta$. By the definition of the norm in Lip X, there exist $x, y \in X$, $x \neq y$ such that

$$(F_1(y) - F_1(x))/\| y - x \| \geqslant \| f \|_Y + \epsilon, \tag{16}$$

where $0 < \epsilon < \delta$. By definition (15) of F_1, for $0 < \eta < \epsilon \| x - y \|$, there exist G_1, $G_2 \in E_Y{}^c(f)$ such that $G_1(x) < F_1(x) + \eta$ and $G_2(y) < F_1(y) + \eta$. The set $E_Y{}^c(f)$ being downward directed (Lemma 4), there exists $G_3 \in E_Y{}^c(f)$ such that $G_3 \leqslant G_1$ and $G_3 \leqslant G_2$. Consequently

$$F_1(x) \leqslant G_3(x) < F_1(x) + \eta$$

and

$$F_1(y) \leqslant G_3(y) < F_1(y) + \eta$$

or, equivalently,

$$0 \leqslant G_3(x) - F_1(x) < \eta,$$

and

$$0 \leqslant G_3(y) - F_1(y) < \eta.$$

From these inequalities one obtains

$$G_3(x) - F_1(x) - (G_3(y) - F_1(y)) \leqslant G_3(x) - F_1(x) < \eta,$$

so that

$$G_3(y) - G_3(x) > F_1(y) - F_1(x) - \eta. \tag{17}$$

Taking into account (16) and (17)

$$\frac{G_3(y) - G_3(x)}{\| y - x \|} > \frac{F_1(y) - F_1(x)}{\| y - x \|} - \frac{\eta}{\| y - x \|}$$

$$> \| f \|_Y + \epsilon - \frac{\eta}{y - x} > \| f \|_Y .$$

But then $\| G_3 \| > \| f \|_Y$, in contradiction to $G_3 \in E_Y{}^c(f)$.
Theorem 3 is proved.

Remark. Let $X = R$ and $Y = [a, b]$, $0 \in Y$. For $f \in K_Y$, let

$$m_1 = \min(| f'(a + 0)| , | f'(b - 0)|)$$

and

$$m_2 = \max(| f'(a + 0)| , | f'(b - 0)|).$$

Then the minimal and maximal convex norm-preserving extensions F_1 and F_2, respectively, of f, are given by

$$\begin{aligned} F_i(x) &= f(x) && \text{for } x \in [a, b], \\ &= f(x) - m_i(x - a) && \text{for } x \in]-\infty, a[, \\ &= f(x) + m_i(x - b) && \text{for } x \in]b, +\infty[; \end{aligned}$$

$i = 1, 2$.

Let now X be a normed linear space, Y a convex subset of X such that $0 \in Y$, and Z a nonvoid bounded subset of X.

Consider the space

$$\mathrm{Lip}_0(X, Z) = \{ f |_Z : f \in \mathrm{Lip}_0 X\},$$

normed by the uniform norm

$$\| f |_Z \|_u = \sup\{| f |_Z (x)| : x \in Z\}.$$

Consider the following problem:

(A) *For $f \in K_X$, find two elements g_* and g^* in $P_{Y_c^\perp}(f)$ such that*

$$\|f|_Z - g_*|_Z\|_u = \inf\{\|f|_Z - g|_Z\|_u : g \in P_{Y_c^\perp}(f)\}$$

and

$$\|f|_Z - g^*|_Z\|_u = \sup\{\|f|_Z - g|_Z\|_u : g \in P_{Y_c^\perp}(f)\}.$$

THEOREM 5. *Problem* (A) *has a solution for all $f \in K_X$.*

Proof. By Theorem 2(b) every g in $P_{Y_c^\perp}(f)$ has the form $g = f - F$ for a convex norm-preserving extension F of $f|_Y$. By Theorem 3, there exist two convex norm-preserving extensions F_1 and F_2 of $f|_Y$ such that

$$F_1(x) \leqslant F(x) \leqslant F_2(x),$$

for all $x \in X$, i.e.,

$$f(x) - g_1(x) \leqslant f(x) - g(x) \leqslant f(x) - g_2(x),$$

for all $x \in X$, where $g_i = f - F_i$, $i = 1, 2$. Therefore

$$\begin{aligned}\min(\|f|_Z - g_1|_Z\|_u, \|f|_Z - g_2|_Z\|_u) &\leqslant \|f|_Z - g|_Z\|_u \\ &\leqslant \max(\|f|_Z - g_1|_Z\|_u, \|f|_Z - g_2|_Z\|_u).\end{aligned}$$

It follows that a solution of Problem (A) is given by $g_* = g_i$ and $g^* = g_j$, where $i, j \in \{1, 2\}$ are such that

$$\|f|_Z - g_i|_Z\|_u = \min(\|f|_Z - g_1|_Z\|_u, \|f|_Z - g_2|_Z\|_u)$$

and

$$\|f|_Z - g_j|_Z\|_u = \max(\|f|_Z - g_1|_Z\|_u, \|f|_Z - g_2|_Z\|_u).$$

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